Note Title 2/22/2008

Definition of a unity operator:

$$\boxed{\sum_n |n\rangle\langle n| = \hat{1}}$$ very useful!

Let's check it:

$$\hat{1}\,|m\rangle = |m\rangle$$

$$\underbrace{\sum_n |n\rangle\langle n|}_{=\hat{1}}\,m\rangle = \sum \delta_{nm}|n\rangle = |m\rangle \quad \checkmark$$

$$\langle n|m\rangle = \delta_{nm}$$

Expectation value (Average)

$$\langle A\rangle = \int \psi^* \hat{A}\,\psi\, d^3r = \langle \psi|\hat{A}|\psi\rangle$$

Expectation value of the Hamiltonian:

$$\langle \hat{H}\rangle = \langle \psi_n|\hat{H}|\psi_n\rangle = E_n$$

Commutator

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

Anticommutator

$$\{\hat{A},\hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$$

Commutator properties:

1. $$[A, B] = -[B, A]$$

2. $$[A, B+C] = A(B+C) - (B+C)A$$
$$= AB - BA + AC - CA$$
$$= [A, B] + [A, C]$$

3. $$[A, BC] = B[A, C] - [A, B]C$$

4. $$[A, B]^\dagger = (AB - BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = -[A^\dagger, B^\dagger]$$

if A, B are Hermitian: $[A, B]^\dagger = -[A, B]$

Measurement

$$\hat{A}\,|n\rangle = a_n\,|n\rangle$$

operator eigenvector eigenvalue

Commutator of x & p:

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x}$$

$$[\hat{x}, \hat{p}]\,|\psi\rangle = x\left(-i\hbar\frac{\partial}{\partial x}\right)|\psi\rangle - \left(-i\hbar\frac{\partial}{\partial x}\right)x\,|\psi\rangle$$

$$= \cancel{x\left(-i\hbar\frac{\partial}{\partial x}\right)|\psi\rangle} + i\hbar\,|\psi\rangle + \cancel{x\left(i\hbar\frac{\partial}{\partial x}\right)|\psi\rangle}$$

$$= i\hbar\,|\psi\rangle$$

$$\Rightarrow \boxed{[\hat{x}, \hat{p}_x] = i\hbar}$$ very useful

Note that $[\hat{x}, \hat{P}_y] = [\hat{x}, \hat{P}_z] = 0$

Expectation values

$$\langle \hat{r} \rangle = \int \psi^*(r)\, r\, \psi(r)\, d^3r = \langle \psi | r | \psi \rangle$$

$$\langle \hat{T} \rangle = \langle \psi | \hat{T} | \psi \rangle = \langle \psi | -\frac{\hbar^2}{2m} \nabla^2 | \psi \rangle$$

$$= -\frac{\hbar^2}{2m} \langle \psi | \nabla^2 | \psi \rangle$$

$$\langle \hat{V} \rangle = \langle \psi | \hat{V} | \psi \rangle$$

Time dependence of expectation value $\frac{d\langle \hat{A} \rangle}{dt} = ?$

$$\hat{H} |\psi\rangle = E |\psi\rangle$$

If time dependent:

$$\hat{H} |\psi\rangle = i\hbar \frac{\partial}{\partial t} |\psi\rangle \rightarrow |\frac{\partial \psi}{\partial t}\rangle = \frac{-i}{\hbar} \hat{H} |\psi\rangle$$

$$\langle \frac{\partial \psi}{\partial t} | = \frac{i}{\hbar} \langle \psi | \hat{H}$$

$$\frac{d\langle \hat{A} \rangle}{dt} = \frac{d}{dt} \langle \psi | \hat{A} | \psi \rangle$$

$$= \langle \frac{\partial \psi}{\partial t} | A | \psi \rangle + \langle \psi | \frac{\partial A}{\partial t} | \psi \rangle + \langle \psi | A | \frac{\partial \psi}{\partial t} \rangle$$

$$= \frac{i}{\hbar} \langle \psi | HA | \psi \rangle - \frac{i}{\hbar} \langle \psi | AH | \psi \rangle + \langle \psi | \frac{\partial A}{\partial t} | \psi \rangle$$

$$= \frac{i}{\hbar} \langle \psi | HA - AH | \psi \rangle + \langle \psi | \frac{\partial A}{\partial t} | \psi \rangle$$

$$= \frac{i}{\hbar} \langle \psi | [H, A] | \psi \rangle + \langle \frac{\partial A}{\partial t} \rangle$$

$$\frac{d}{dt} \langle \hat{A} \rangle = \frac{i}{\hbar} \langle [H, A] \rangle + \langle \frac{\partial A}{\partial t} \rangle$$

So if $\hat{A}$ is time independent:

$$\frac{d}{dt} \langle \hat{A} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{A}] \rangle$$

Easier to remember if write it in this form:

$$\boxed{[H, A] = -i\hbar \frac{dA}{dt}}$$